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TWO MOST NONDETERMINISTIC PROGRAMS

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Abstract. The computations evoked by starting a program can differ with respect to termination and—in case of termination—final state. This paper contains two examples, together with formal proofs, of programs with different classes of possible computations for different initial states.

Introduction

For any program S and predicate R on the state space of S, $wp(S, R)$ and $wlp(S, R)$ are predicates on the state space of S, and allow the following operational interpretation:

$wp(S, R)$ ≡ "starting S always leads to a terminating computation, and the final state satisfies R",

and

$wlp(S, R)$ ≡ "whenever starting S leads to a terminating computation, the final state satisfies R".

In particular,

$wp(S, true)$ ≡ "starting S always leads to a terminating computation",

and

$wlp(S, \neg R)$ ≡ "starting S never leads to a computation that terminates in a state satisfying R".

When the computation evoked by starting S is not uniquely determined by the initial state, S is called nondeterministic; the words 'always', 'whenever', and 'never' are understood to range over all computations to which starting S can lead.

A formal definition of the predicate transformers *wp* and *wlp* is given in the Appendix; the operational interpretation given above is not used in the remainder of this article.

1. Subject of the paper

From the three predicates $wp(S, true)$, $wlp(S, R)$, and $wlp(S, \neg R)$, we can form eight predicates, viz.

$$\neg wp(S, true) \wedge \neg wlp(S, R) \wedge \neg wlp(S, \neg R), \tag{0}$$

$$\neg wp(S, true) \wedge \neg wlp(S, R) \wedge wlp(S, \neg R), \tag{1}$$

$$\neg wp(S, true) \wedge wlp(S, R) \wedge \neg wlp(S, \neg R), \tag{2}$$

$$\neg wp(S, true) \wedge wlp(S, R) \wedge wlp(S, \neg R), \tag{3}$$

$$wp(S, true) \wedge \neg wlp(S, R) \wedge \neg wlp(S, \neg R), \tag{4}$$

$$wp(S, true) \wedge \neg wlp(S, R) \wedge wlp(S, \neg R), \tag{5}$$

$$wp(S, true) \wedge wlp(S, R) \wedge \neg wlp(S, \neg R), \tag{6}$$

$$wp(S, true) \wedge wlp(S, R) \wedge wlp(S, \neg R), \tag{7}$$

of which in any point of the state space of S exactly one holds. Since the Law of the Excluded Miracle (see Appendix) precludes the existence of a state satisfying (7), in any point of the state space of S exactly one of (0) through (6) holds. In this paper, we present two examples of a program S with a predicate R, for which each of (0) through (6) is satisfied in exactly one point of the state space of S.

Some of the proofs in this paper use the notions of strongest resp. weakest solutions of an equation; these notions are defined and exemplified in the Appendix.

Notational note. Surrounding a predicate by square brackets means universally quantifying it over the state space.

2. Design decisions

Obviously, the state space of our examples has exactly seven points; our programs have just one variable, x say, taking integer values at least 0 and at most 6. Furthermore, we require that for $0 \leq k \leq 6$, predicate (k) be satisfied in the point $x = k$. It then suffices to show that, with A, B, and C defined by

$$[A \equiv x = 4 \vee x = 5 \vee x = 6],$$

$$[B \equiv x = 2 \vee x = 3 \vee x = 6],$$

$$[C \equiv x = 1 \vee x = 3 \vee x = 5],$$

our examples satisfy

$$[wp(S, true) \equiv A], \tag{8}$$

$$[wlp(S, R) \equiv B], \tag{9}$$

$$[wlp(S, \neg R) \equiv C]. \tag{10}$$

Note that by our choice of state space we have

$$[\neg A \vee \neg B \vee \neg C]. \tag{11}$$

3. First example

With programs $S0$, $S1$, $S2$, and predicate R such that

$$[\neg wp(S0, true) \wedge wlp(S0, R) \wedge wlp(S0, \neg R)], \tag{12}$$

$$[wp(S1, true) \wedge \neg wlp(S1, R) \wedge wlp(S1, \neg R)], \tag{13}$$

$$[wp(S2, true) \wedge wlp(S2, R) \wedge \neg wlp(S2, \neg R)], \tag{14}$$

the program

S: **if** $\neg A \rightarrow S0 \,\square\, \neg B \rightarrow S1 \,\square\, \neg C \rightarrow S2$ **fi**

satisfies (8), (9), and (10).

Proof.

$wp(S, true)$

$=$ {definition of wp (see Appendix)}

$(\neg A \vee \neg B \vee \neg C) \wedge$

$(A \vee wp(S0, true)) \wedge (B \vee wp(S1, true)) \wedge (C \vee wp(S2, true))$

$=$ {(11), (12), (13), and (14)}

A,

which proves (8).

$wlp(S, R)$

$=$ {definition of wlp (see Appendix)}

$(A \vee wlp(S0, R)) \wedge (B \vee wlp(S1, R)) \wedge (C \vee wlp(S2, R))$

$=$ {(12), (13), and (14)}

B,

which proves (9).

The proof of (10) is similar. □

With $x = 6$ for R, the programs

$S0$: **do** $true \rightarrow skip$ **od**,

$S1$: $x := 5$,

$S2$: $x := 6$

satisfy (12), (13), and (14) respectively.

Proof. For any predicate P, $wp(S0, P)$ and $wlp(S0, P)$ are, by definition (see Appendix), the strongest and the weakest solution respectively of the equation

$$X: \quad [X \equiv (\mathit{false} \vee X) \wedge (\mathit{true} \vee P)],$$

which has *false* as its strongest solution and *true* as its weakest.
Hence,

$$[wp(S0, \mathit{true}) \equiv \mathit{false}], \quad [wlp(S0, R) \equiv \mathit{true}] \quad \text{and} \quad [wlp(S0, \neg R) \equiv \mathit{true}],$$

which proves (12).

The definitions of *wp* and *wlp* (see Appendix), together with the choice of R, immediately imply (13) and (14). □

4. Second example

With again $x = 6$ for R, and with D defined by

$$[D \equiv x = 5 \vee x = 6],$$

the following program

S: **do** $\neg(A \vee D) \rightarrow \mathit{skip}$ □ $\neg(B \vee D) \rightarrow x := 5$ □ $\neg(C \wedge D) \rightarrow x := 6$ **od**

also satisfies (8), (9), and (10).

Proof. For any predicate P, $wp(S, P)$ and $wlp(S, P)$ are, by definition (see Appendix), the strongest and the weakest solution respectively of the equation

$$X: \quad [X \equiv (A \vee D \vee X) \wedge (B \vee D \vee X_5^x) \wedge (C \vee D \vee X_6^x) \wedge (\neg(A \vee D) \vee \neg(B \vee D) \vee \neg(C \vee D) \vee P)],$$

which on account of (11) reduces to

$$X: \quad [X \equiv (A \vee D \vee X) \wedge (B \vee D \vee X_5^x) \wedge (C \vee D \vee X_6^x) \wedge (\neg D \vee P)]. \qquad (15)$$

Since $[D_5^x \equiv \mathit{true}]$ and $[D_6^x \equiv \mathit{true}]$, any solution X of (15) satisfies $[X_5^x \equiv P_5^x]$ and $[X_6^x \equiv P_6^x]$, as can be seen by substituting 5 and 6 respectively for x in both sides of (15).

Hence, any solution of (15) also solves

$$X: \quad [X \equiv (A \vee D \vee X) \wedge (B \vee D \vee P_5^x) \wedge (C \vee D \vee P_6^x) \vee (\neg D \vee P)]. \qquad (16)$$

Conversely, any solution X of (16) satisfies $[X_5^x \equiv P_5^x]$ and $[X_6^x \equiv P_6^x]$, and hence solves (15). Therefore, (15) and (16) have the same set of solutions.

Hence, $wp(S, \mathit{true})$ is the strongest solution of (16) with *true* for P, i.e. the strongest solution of

$$X: \quad [X \equiv A \vee D \vee X],$$

which is $A \vee D$. Since $[A \vee D \equiv A]$, this proves (8).

Furthermore, $wlp(S, R)$ is the weakest solution of (16) with R for P, i.e., since $[R_5^x \equiv false]$ and $[R_6^x \equiv true]$, the weakest solution of

$$X\colon\ [X \equiv (A \vee D \vee X) \wedge (B \vee D) \wedge (\neg D \vee R)],$$

which is $(B \vee D) \wedge (\neg D \vee R)$. Since $[(B \vee D) \wedge (\neg D \vee R) \equiv B]$, this proves (9).

The proof of (10) is analogous to that of (9). □

Appendix.

A.1. Formal definition of wp and wlp

$[wp(skip, P) \equiv P]$,

$[wlp(skip, P) \equiv P]$,

$[wp(x := E, P) \equiv def(E)$ **cand** $P_E^x]$,

$[wlp(x := E, P) \equiv \neg def(E)$ **cor** $P_E^x]$,

$[wp(\mathrm{IF}, P) \equiv (\mathbf{E}i :: Bi) \wedge (\mathbf{A}i :: \neg Bi \vee wp(Si, P))]$,

$[wlp(\mathrm{IF}, P) \equiv (\mathbf{A}i :: \neg Bi \vee wlp(Si, P))]$,

$[wp(\mathrm{DO}, P) \equiv$ the strongest solution of the equation

$X\colon\ [X \equiv (\mathbf{A}i :: \neg Bi \vee wp(Si, X)) \wedge ((\mathbf{E}i :: Bi) \vee P)]$

],

$[wlp(\mathrm{DO}, P) \equiv$ the weakest solution of the equation

$X\colon\ [X \equiv (\mathbf{A}i :: \neg Bi \vee wlp(Si, X)) \wedge ((\mathbf{E}i :: Bi) \vee P)]$,

],

where P is a predicate over the appropriate state space, IF stands for

if $B0 \to S0 \,\square \cdots \square\, Bn \to Sn$ **fi**,

and DO stands for

do $B0 \to S0 \,\square \cdots \square\, Bn \to Sn$ **od**.

For any program S, and predicates P and Q over the state space of S, we have

$[wp(S, false) \equiv false]$ (Law of the Excluded Miracle),

$[wp(S, P) \equiv wp(S, true) \wedge wlp(S, P)]$,

$[wlp(S, P \wedge Q) \equiv wlp(S, P) \wedge wlp(S, Q)]$.

Proofs of these and other properties of *wp* and *wlp* can be found in [1].

A.2. Strongest and weakest solutions

A predicate is the strongest solution of an equation if it solves the equation and implies any other solution of the equation.

A predicate is the weakest solution of an equation if it solves the equation and is implied by any other solution of the equation.

For example, the strongest solution of the equation

$$X\colon\ [X \equiv Y \vee X]$$

is Y, and the weakest solution of the equation

$$X\colon\ [X \equiv (Y \vee X) \wedge Z]$$

is Z.

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Reference

[1] E.W. Dijkstra, Lecture notes "Predicate transformers" (Draft), EWD835, Eindhoven University of Technology (1982).